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1981 J. Phys. A: Math. Gen. 14 471

(http://iopscience.iop.org/0305-4470/14/2/022)

J. Phys. A: Math. Gen. 14 (1981) 471–477. Printed in Great Britain

New topological indices in SO(3) Einstein–Yang–Mills theory

R F Picken

Department of Physics, Queen Mary College, Mile End Road, London E1 4NS, England

Received 15 May 1980, in final form 18 August 1980

Abstract. The topological classification of solutions to the Euclideanised SO(3) Einstein–Yang–Mills equations is discussed. As well as the usual Pontriagin index, two new two-valued indices, associated with the second Stiefel–Whitney class, are required to specify the bundles. The Charap–Duff solution is classified in this framework.

1. Introduction

Because of the current interest in Yang–Mills instantons, several authors (Charap and Duff 1977, Boutaleb Joutei and Chakrabarti 1979, Duff and Madore 1979, Pope and Yuille 1978) have examined the Yang–Mills system in a curved space–time background, but have considered only SU(2) instantons or pseudoparticles. In this paper we change the gauge group to SO(3) and show that one obtains, as well as the usual Pontriagin index, new two-valued topological indices not present in flat space–time or in an SU(2) theory.

In § 2 we describe the Euclideanised Einstein–Yang–Mills system, with a Schwarzschild background geometry. The classification problem is equivalent to classifying principal G-bundles over $S^2\times S^2$. In § 3 we describe this classification when $G=$ SO(3). In § 4 we classify a subset of the possible bundles in terms of maps from the space–time boundary to SO(3). In § 5 we calculate some potentials at spatial infinity and classify the Charap–Duff pseudoparticle solution. Finally § 6 contains some conclusions.

2. Yang–Mills pseudoparticles in curved space–time

We consider (anti-) self-dual solutions to the Euclideanised Einstein–Yang–Mills equations. The Yang–Mills field

$$F_{\mu\nu}=F^{i}_{\mu\nu}\lambda_i,$$

where λ_i are the generators of G, satisfies the (anti-)self-duality condition

$$F_{\mu\nu}=\pm{}^*F_{\mu\nu}\equiv\pm\tfrac{1}{2}\eta_{\mu\nu\alpha\beta}F^{\alpha\beta},$$

0305-4470/81/020471+07\$01.50

where $\eta_{\mu\nu\alpha\beta} = \sqrt{g}\epsilon_{\mu\nu\alpha\beta}$, and hence (because of the Bianchi identities) the Yang–Mills equations

$$g^{\mu\rho}\nabla_\rho F_{\mu\nu} = 0,$$

where ∇ is the derivative w.r.t. both the gravitational and Yang–Mills connections.

As the energy-momentum tensor $T_{\mu\nu}$ vanishes for (anti-)self-dual configurations, the metric $g_{\mu\nu}$ can be any solution to the vacuum Einstein equations (Charap and Duff 1977). We choose a Schwarzschild black hole background as we wish to examine the effects of a non-trivial space–time topology. The Schwarzschild line element is given in Kruskal coordinates $(\lambda, \eta, \theta, \phi)$ by

$$\begin{aligned} ds^2 &= g_{\mu\nu}\, dx^\mu\, dx^\nu \\ &= \left(\frac{1}{4m}\right)^3 \frac{\exp(-r/2m)}{r}(d\lambda^2 + d\eta^2) + r^2(d\theta^2 + \sin^2\theta\, d\phi^2), \end{aligned}$$

where

$$\lambda = \exp(r/4m)(1/4m)(r-2m)^{1/2}\sin(\tau/4m),$$

$$\eta = \exp(r/4m)(1/4m)(r-2m)^{1/2}\cos(\tau/4m).$$

Here τ is the $8\pi m$-periodic time coordinate. Hence the topology of the space–time is $R^2 \times S^2$ (Hawking 1977, Duff and Madore 1979).

For convergence of the action integral, $F_{\mu\nu}$ must tend to zero as $r \to \infty$. This boundary condition can be replaced by the geometric constraint, that we seek instead solutions on the compactified space–time $S^2 \times S^2$ which is obtained from $R^2 \times S^2$ by shrinking the hypersurface $r = \infty$ to a point. In mathematical terms then, we are considering principal G bundles on a base space with topology $S^2 \times S^2$, whose curvature $F_{\mu\nu}$ satisfies the Yang–Mills equations.

3. The topological classification of principal SO(3) bundles over $S^2 \times S^2$

The group of isomorphism classes of principal G bundles on S^n, denoted $B_G(S^n)$, is isomorphic to the homotopy group $\Pi_{n-1}(G)$ of homotopy-equivalence classes of continuous mappings $S^{n-1} \to G$. This is because S^n can be covered by two open disc-like regions, D_n^1 and D_n^2, on each of which the bundle is necessarily trivial and the topological classification is given by a map from the overlap region $D_n^1 \cap D_n^2$ to G, which reduces to a map $S^{n-1} \to G$. Thus, for instance, SU(2) instantons on compactified flat space–time S^4 are classified by $\Pi_3(\mathrm{SU}(2)) = \mathbb{Z}$.

SO(3), the group of three-dimensional rotations, can be topologised in the following manner. Map the rotation $(\hat{n}, \alpha)$ (about the axis $\hat{n}$ by an angle $\alpha \leq \pi$) to the vector $\alpha\hat{n}$. The resulting space is a 3-disc of radius π, with opposite points on the boundary identified as $(\hat{n}, \pi) = (-\hat{n}, \pi)$. The topology of SO(3) is therefore RP^3, real projective 3-space, which can also be thought of as a 3-sphere S^3 with antipodal points identified. The low ($n \leq 3$) homotopy groups of SO(3) are $\Pi_1(\mathrm{SO}(3)) = \mathbb{Z}_{/2}$, $\Pi_2(\mathrm{SO}(3)) = 0$ and $\Pi_3(\mathrm{SO}(3)) = \mathbb{Z}$. (The statement $\Pi_1(\mathrm{SO}(3)) = \mathbb{Z}_{/2}$ implies that there are non-trivial loops on SO(3) which cannot be continuously deformed to a point. An example of such a loop is the z axis $(-\pi \leq z \leq \pi)$ linking the (identical) points $\pi(0, 0, 1)$ and $\pi(0, 0, -1)$.)

We now cover $S^2 \times S^2$ with two overlapping regions on which we can analyse the bundle using the above information. Firstly we explain this covering in an easier lower-dimensional case, namely the torus $S^1 \times S^1$ which can be thought of as a rectangle with opposite edges identified. We cover the torus with two regions A and B as in figure 1. A is an open 2-disc D^2 bounded by a circle S^1. B is also bounded by S^1 and is a neighbourhood of a figure-of-eight denoted $S^1 \vee S^1$ ($X \vee Y$ is the one-point union of X and Y) or, more precisely, $S^1 \vee S^1$ is a deformation retract of B. In an analogous fashion we can cover $S^2 \times S^2$ with two regions A and B, where A is a 4-disc D^4 with boundary S^3 and B is a neighbourhood of $S^2 \vee S^2$($S^2 \vee S^2$ is a deformation retract of B) with boundary S^3 also.

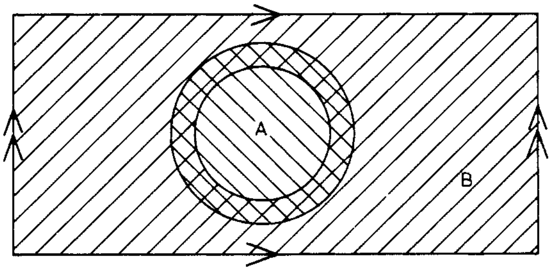

Figure 1.

By mapping A $= D^4$ with its boundary S^3 into S^4 by identifying S^3 to a point, we can pull back bundles over S^4 which are characterised by $\Pi_3(\mathrm{SO}(3)) = \mathbb{Z}$. Furthermore, as $S^2 \vee S^2$ is a deformation retract of B, the bundles on B are characterised by

$$\Pi_1(\mathrm{SO}(3)) \oplus \Pi_1(\mathrm{SO}(3)) = \mathbb{Z}_{/2} \oplus \mathbb{Z}_{/2}.$$

Finally, the overlap region can be retracted to S^3 on which the bundle is necessarily trivial as $\Pi_2(\mathrm{SO}(3)) = 0$. Thus $B_{\mathrm{SO}(3)}(S^2 \times S^2)$ is isomorphic to $\mathbb{Z} \oplus \mathbb{Z}_{/2} \oplus \mathbb{Z}_{/2}$, and hence bundles are classified by three numbers: an integer which is the winding number or Pontriagin index, and two two-valued indices corresponding to the second Stiefel–Whitney class.

This is in agreement with more general results obtained by Avis and Isham (1979), who state that for simply connected, four-dimensional manifolds the SO(3) bundles are classified by a pair of classes

$$(p_1, w_2) \in H^4(M, \mathbb{Z}) \oplus H^2(M, \mathbb{Z}_{/2}),$$

where p_1 is the (first) Pontriagin class and w_2 the second Stiefel–Whitney class. This pair must satisfy the general relation

$$p_1 \bmod 2 = w_2 \cup w_2,$$

where $\cup$ is the cup product. In our case there are no non-vanishing square elements of $H^2(M, \mathbb{Z}_{/2})$ and the condition becomes $p_1 \bmod 2 = 0$. In other words the Pontriagin index must be an even number. This is not surprising, because the Pontriagin index comes from the winding number of SO(3) bundles over (D^4, S^3) pulled back from (S^4, pt) and these are classified by maps $S^3 \to \mathrm{SO}(3)$, which are necessarily of even degree as the map factorises into $S^3 \to \mathrm{SU}(2) \to \mathrm{SO}(3)$, and $\mathrm{SU}(2) \to \mathrm{SO}(3)$ (the projection map) is of degree 2.

4. Classification of bundles trivial on the (θ, ϕ) sphere

We now examine our case from a slightly different point of view, paraphrasing an argument frequently used in the flat space–time instanton case. As we require the action integral to be finite, the field strength $F_{\mu\nu}$ must vanish at $r=\infty$, and therefore A_μ, the Yang–Mills potential, must be a gauge transform of zero at $r=\infty$: $A_\mu(r=\infty)=g^{-1}\partial_\mu g$. Thus it would seem that solutions are characterised by maps from the boundary $S^2\times S^1$, parametrised by $(\theta, \phi; \xi\equiv\tau/4m)$ to the group G. This is not the case, however: we are, in effect, covering $S^2\times S^2$ with two open regions $S^2\times E_+$ and $S^2\times E_-$, where E_+ is the 2-disc $r<\infty$ and E_- is a small two-dimensional neighbourhood of $r=\infty$. The overlap region $S^2\times E_+\cap S^2\times E_-$ can be retracted to $S^2\times S^1$. However, bundles are characterised by maps from this overlap region to G only if the bundle is trivial on $S^2\times E_+$ and $S^2\times E_-$. If $G=\mathrm{SO}(3)$ this does not necessarily hold, as SO(3) bundles on $S^2\times E_\pm$ are classified by $\Pi_1(\mathrm{SO}(3))=\mathbb{Z}_{/2}\neq 0$. However, we can classify bundles *which are trivial on the* (θ, ϕ) *sphere* by maps $S^2(\theta, \phi)\times S^1(\xi)\to \mathrm{SO}(3)$. We proceed to examine the homotopy type of such maps to obtain the classification and behaviour at infinity of bundles trivial on the (θ, ϕ) sphere. By our previous analysis we would expect this homotopy group to be isomorphic to

$$\mathbb{Z}\oplus\mathbb{Z}_{/2}\oplus 0\cong\mathbb{Z}\oplus\mathbb{Z}_{/2}.$$

There are two ways of constructing non-trivial maps from $S^2\times S^1$ to $\mathrm{SO}(3)=RP^3$. Firstly, recalling that $\Pi_1(RP^3)=\mathbb{Z}_{/2}$, we can trivialise the S^2 part and map $S^2\times S^1\to S^1\to RP^3$ along a representative loop of the non-trivial class of $\Pi_1(RP^3)$. Secondly, one can map $S^2\times S^1$ non-trivially onto S^3 and then project onto RP^3. The basic non-trivial map $S^2\times S^1\to S^3$ involves mapping an open 3-disc-like region on $S^2\times S^1$ to $S^3\backslash\{N \text{ pole}\}$ and mapping the rest of $S^2\times S^1$ to the N pole. At the very least, one 1-cycle and one 2-cycle must be mapped to the N pole (or to a set that can be contracted to a single point). One still has the option of performing an n-fold automorphism of S^3 before dropping to RP^3, giving a map of degree $2n$.

Regarding the above two maps as generators of the homotopy group of maps $S^2\times S^1\to\mathrm{SO}(3)$, the resulting group is $\mathbb{Z}\oplus\mathbb{Z}_{/2}$ as expected. Thus the maps are characterised by an even integer, the degree or Pontriagin index, and another two-valued number (0 or 1) corresponding to the $\mathbb{Z}_{/2}$ subgroup. Maps in any class can be obtained by group multiplication of the image points, i.e.

$$g(2n, 0)=g(2, 0)^n,$$
$$g(2n, 1)=g(2n, 0)g(0, 1).$$

5. The potentials at infinity for bundles trivial on the (θ, ϕ) sphere

We now proceed to exhibit explicit maps and calculate the corresponding potentials at infinity.

Let $S^2\times S^1$ be parametrised by $(\hat{X}; \xi)$, where $\hat{X}$ is a unit vector determined by (θ, ϕ) through the correspondence

$$\hat{X}=(x, y, z)=(\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$$

and $\xi=\tau/4m$ is a 2π-periodic variable parametrising S^1.

A typical (0, 1) map is then given by

$$g(0,1)(\hat{X};\xi)=\exp(\xi\hat{X}'\cdot\boldsymbol{\lambda}),$$

where $\boldsymbol{\lambda}=(\lambda_1,\lambda_2,\lambda_3)$ are the generators of SO(3), and $\hat{X}'$ is an arbitrary constant unit vector (in our example from § 3 $\hat{X}'=(0,0,1)$).

A typical (2, 0) map is given by

$$g(2,0)(\hat{X};\xi)=\exp(\xi\hat{X}\cdot\boldsymbol{\lambda})\exp(-\xi\hat{X}'\cdot\boldsymbol{\lambda}),$$

where $\hat{X}'$ is again an arbitrary fixed direction. This is so because:

(*a*) $g(2,0)$ maps the two-cycle $\xi=0$ and the one-cycle $\hat{X}=\hat{X}'$ to 1.

(*b*) $g(2,0)$ is onto, and the pre-image of $g'\in \operatorname{Im} g(2,0)$, $g'\neq 1$ consists of precisely two-points; hence the degree of $g(2,0)$ is 2.

To show (*b*) requires a certain amount of calculation, and we refer the reader to the Appendix.

By multiplying these two maps, we get a (2, 1) map:

$$g(2,1)=g(2,0)g(0,1)=\exp(\xi\hat{X}\cdot\boldsymbol{\lambda})\exp(-\xi\hat{X}'\cdot\boldsymbol{\lambda})\exp(\xi\hat{X}''\cdot\boldsymbol{\lambda}).$$

By choosing $\hat{X}'=\hat{X}''$ we have

$$g(2,1)=\exp(\xi\hat{X}\cdot\boldsymbol{\lambda}).$$

These maps give rise to the following potentials at infinity ($A=A_\mu\,\mathrm{d}X^\mu$):

$$A^{(0,1)}=(\hat{X}'\cdot\boldsymbol{\lambda})\,\mathrm{d}\xi,$$
$$A^{(2,0)}=\exp(\xi\hat{X}''\cdot\boldsymbol{\lambda})A^{(2,1)}\exp(-\xi\hat{X}''\cdot\boldsymbol{\lambda})-(\hat{X}''\cdot\boldsymbol{\lambda})\,\mathrm{d}\xi,$$
$$A^{(2,1)}=[\hat{X}\,\mathrm{d}\xi-(1-\cos\xi)(\hat{X}\times\mathrm{d}\hat{X})+\sin\xi\,\mathrm{d}\hat{X}]\cdot\boldsymbol{\lambda}.$$

An anti-self-dual SU(2) pseudoparticle (Charap and Duff 1977) is given in rectangular coordinates by

$$A=-\frac{m}{r^2}\frac{X^i}{r}\frac{\sigma^i}{2\mathrm{i}}\,\mathrm{d}\tau+(1-\alpha)\epsilon^{ijk}\frac{X^i}{r^2}\frac{\sigma^j}{2\mathrm{i}}\,\mathrm{d}X^\kappa,$$

where $\alpha=(1-2m/r)^{1/2}$. Its Pontriagin index is found to be 1.

By replacing $\sigma^i/2\mathrm{i}$ by λ_i and performing the gauge transformation

$$A'_\mu=g(2,1)^{-1}A_\mu g(2,1)+g(2,1)^{-1}\partial_\mu g(2,1)$$

to ensure that $A_\tau=0$ at the horizon $r=2m$, we get for A':

$$A'=\left[\left(1-\frac{4m^2}{r^2}\right)\hat{X}\,\mathrm{d}\xi-(1-\alpha\cos\xi)(\hat{X}\times\mathrm{d}\hat{X})+\alpha\sin\xi\,\mathrm{d}\hat{X}\right]\cdot\boldsymbol{\lambda}.$$

As the Pontriagin index is the degree of the map $S^2\times S^1\to G$ and it is 1 for $G=\mathrm{SU}(2)$, it must be 2 for $G=\mathrm{SO}(3)$. Indeed at $r=\infty$, A' takes the form of $A^{(2,1)}$. Also, when we transform A' to angular coordinates we find $A'_\phi=0$ for $\theta=0,\pi$. Therefore A' is regular and hence trivial on the (θ,ϕ) sphere. In conclusion, the anti-self-dual SO(3) Charap–Duff solution is in the (2, 1, 0) sector of $\mathbb{Z}\oplus\mathbb{Z}_{/2}\oplus\mathbb{Z}_{/2}$. Similar results hold for the self-dual case.

6. Conclusions

We have shown that SO(3) pseudoparticles on Euclideanised Schwarzschild space–time have a more complicated topological structure than flat space–time SU(2) instantons which are characterised by a single integer. This is due to an interplay between the new choice of gauge group, SO(3), and the underlying space–time topology $S^2 \times S^2$. (Both SO(3) flat space–time instantons and SU(2) pseudoparticles on Euclideanised Scwarzschild space–time are classified by a single integer.)

Although we have only discussed the Euclideanised Schwarzschild geometry, the topological analysis remains true for an SO(3) gauge theory on any base space whose compactified geometry is homeomorphic to $S^2 \times S^2$.

The Charap–Duff potential only had to be specified on two regions, $r < \infty$ and a neighbourhood of $r = \infty$ (where $A = 0$). In general a potential need not be trivial on one of the S^2 spheres and would have to be specified on four overlapping regions: the above two subdivided into $\theta > 0$ and $\theta < \pi$.

Acknowledgments

I am greatly indebted to C Gordon who first pointed out the connection between $g(2, 1)$ and the Charap–Duff solution, to R Arthan for explaining many of the topological ideas, and especially to Professor J M Charap for supervising this work throughout.

Appendix

We wish to show that $g(2, 0)$ is onto, and that, apart from the point $g = 1$, the pre-image of $g \in \mathrm{SO}(3)$ consists of two points of $S^2 \times S^1$.

It is convenient to factorise $g(2, 0)$ in the following way:

$$\begin{array}{ccc} S^2 \times S^1 & \xrightarrow{g(2,0)} & \mathrm{SO}(3) \\ & {\scriptstyle g'(2,0)} \searrow \quad \nearrow {\scriptstyle h} & \\ & \mathrm{SU}(2) & \end{array}$$

where

$$g'(2, 0) = \exp(-\xi \hat{X}' \cdot \boldsymbol{\sigma}/2\mathrm{i}) \exp(\xi \hat{X} \cdot \boldsymbol{\sigma}/2\mathrm{i})$$

and h is the projection map from SU(2) to SO(3) given by

$$h(Q) = h(-Q) = [a_{ij}] = [\tfrac{1}{2} \mathrm{Tr}(Q\sigma_i Q^\dagger \sigma_j)].$$

It can be checked that

$$h(\exp(\xi \hat{X} \cdot \boldsymbol{\sigma}/2\mathrm{i})) = \exp(\xi \hat{X} \cdot \boldsymbol{\lambda}),$$

and using the identity

$$\tfrac{1}{2} \mathrm{Tr}(XY) = \sum_{i=1}^{3} \tfrac{1}{2} \mathrm{Tr}(\sigma_i X) \cdot \tfrac{1}{2} \mathrm{Tr}(\sigma_i Y)$$

for traceless matrices X, Y, we have

$$h(Q_1Q_2) = h(Q_2)h(Q_1).$$

Hence the diagram commutes.

Thus it remains to show that $g'(2, 0)$ is onto (as h is onto), and that the pre-image of $g \in \mathrm{SU}(2)$ ($g \neq 1$) consists of one point of $S^2 \times S^1$ (as h is 2 : 1).

$g'(2, 0)$ can be written

$$g'(2,0) = 1(\cos^2 \tfrac{1}{2}\xi + z \sin^2 \tfrac{1}{2}\xi) - \mathrm{i}\sigma_1(x \sin \tfrac{1}{2}\xi \cos \tfrac{1}{2}\xi + y \sin^2 \tfrac{1}{2}\xi)$$
$$- \mathrm{i}\sigma_2(y \sin \tfrac{1}{2}\xi \cos \tfrac{1}{2}\xi - x \sin^2 \tfrac{1}{2}\xi) - \mathrm{i}\sigma_3((z-1) \sin \tfrac{1}{2}\xi \cos \tfrac{1}{2}\xi),$$

where we have chosen the arbitrary vector $\hat{X}' = (0, 0, 1)$.

Setting the RHS equal to

$$a1 - \mathrm{i}b\sigma_1 - \mathrm{i}c\sigma_2 - \mathrm{i}d\sigma_3 \qquad (a^2 + b^2 + c^2 + d^2 = 1)$$

and solving the equations, we find

$$\tan \tfrac{1}{2}\xi = (a-1)/d,$$
$$x = (bd - ac + c)/(a-1)$$
$$y = (ab + cd - b)/(a-1)$$
$$z = [a - (a^2 + d^2)]/(a-1) \qquad (a \neq 1).$$

So for any choice of a, b, c, d (apart from $a = 1$) we have a unique choice ($\hat{X}$; ξ) such that

$$g'(2,0)(\hat{X}; \xi) = a1 - \mathrm{i}b\sigma_1 - \mathrm{i}c\sigma_2 - \mathrm{i}d\sigma_3.$$

Hence the result follows.

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